COHERENT RF ERROR STATISTICS*

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ABSTRACT

RF error statistics for power, voltage, and phase are presented for an error component which is coherently related to a desired signal. The error component is assumed to have a constant magnitude and a phase distribution which is equally likely and uniformly distributed from 0 to 360°. The statistics which result have non-zero mean values for power and voltage errors and the standard deviation of the errors differ significantly from those projected from Gaussian statistics.

INTRODUCTION

The accuracy which which RF measurements can be made is a fundamental assessment in microwave applications. Such accuracy assessments are typically addressed through an error budget which combines individual error component values. These individual error components may be divided into two classes, errors which are coherently related to the desired signal component and errors which are incoherently related to the desired signal. The desired signal can be one used for test purposes or a signal which is operationally used. The class of coherently related errors is based on errors which are generated by the desired signal but degrade the performance of the system; examples of this class of error components include VSWR interaction components and multipath errors. The most common example of incoherent errors is thermal noise, and error budgets reflecting the Gaussian statistics associated with additive white noise are commonly used. The statistics for coherent errors have had little attention, and the simple expressions for the statistics of power, voltage, and phase for the coherent errors will be derived.

The theoretical basis between coherent and incoherent error statistics also provides a contrast between the two types of error components. The statistics for incoherent errors are derived under the assumption of a large number of statistically similar components having a zero mean error. The Central Limit Theorem is then invoked to obtain a Gaussian distribution for the collection with zero mean error (Ref. 1). In contrast with the assumptions for incoherent errors, the coherent error case has a limited number of components with significant magnitude rather than a large number of statistically similar components. The resulting statistics for the coherent case can have non-zero mean values and standard deviations which differ significantly from those projected on the basis of Gaussian statistics.

DERIVATION OF THE STATISTICS

The simple phasor diagram, shown in Fig. 1, provides an easy visualization for the coherent error analysis. The true value of the desired signal is represented by unity at a 0° phase angle, while the error component has a relative amplitude a and phase α with respect to the true value. The distribution for the phase of the error component will be assumed to be equally likely and uniformly distributed from 0 to 360°; this assumption for the phase distribution is similar to the incoherent case. These assumptions are used to obtain the statistics for the total power, voltage and phase. Since the true value of the power and voltage is unity and the true value of the phase is 0°, the statistics of the errors can be obtained by subtracting the true value from the statistics of the total value.

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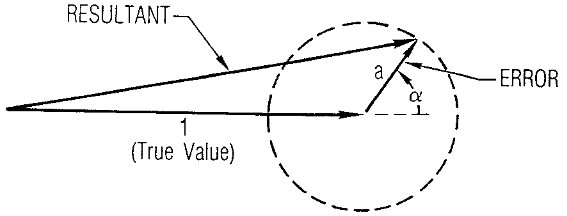

Fig. 1. Phasor Diagram

The statistics derived here are based on a CW variation of the desired signal; more general forms of desired signals can be treated by weighting the amplitude of the error component with the autocorrelation of the desired signal with the appropriate time delay. The peak-to-peak error excursions in the coherent case have been generally used, but the derivation of mean and standard deviation values of the measured quantities are believed to be original in this work. This statistical analysis is necessary in cases in which the phase of the error component is unknown; when both the amplitude and phase of the error component is known, the error can be determined deterministically. Indeed, this calibration of error sources is the basis of the accuracy which can be achieved with modern network analyzers. This statistical analysis fulfills a role in determining measurement accuracy in cases in which the calibration can not be done and in operational systems in which the phase of the error components may be time-varying. The statistical analysis is also a useful way to determine the component performance requirements needed to fulfill the overall system accuracy requirements.

A. Power Statistics

The statistics for the power can be easily derived in closed form. Referring to Fig. 1, the power can be expressed as

$$P = 1 + a^2 + 2a\cos\alpha \qquad (1)$$

The familiar peak-to-peak variation of the power is obtained by setting α to 0^o and 180^o. The mean power level is given by

$$E_P = (1/2\pi)\int_0^{2\pi} P\, d\alpha = 1 + a^2 \qquad (2)$$

The variance of the power is given by

$$V_P = (1/2\pi)\int_0^{2\pi} (P - E_P)^2 d\alpha = 2a^2 \qquad (3)$$

and the corresponding standard deviation is given by

$$\sigma_P = \sqrt{2}\ a \qquad (4)$$

B. Voltage Statistics

The resultant voltage, as seen from Fig. 1, is given by the square root of the power expression given in eq. 1. The mean value of the voltage is given by

$$E_V = (1/2\pi)\int_0^{2\pi} V\, d\alpha = (2/\pi)(1+a)\ E(4a/(1+a)^2) \qquad (5)$$

where $E(X)$ is the complete elliptic integral of the second kind. While the elliptic integrals are tabulated functions, a simple analytic approximation would be useful for calculation purposes. Several polynominal approximations for the elliptic function were investigated. A convenient form for the mean value of the voltage is given by

$$E_V \approx 1 + a^2/4 + a^4/64 \qquad (6)$$

The mean error in the voltage is obtained by subtracting 1 from the expected value of the resultant voltage.

The variance of the voltage can be derived easily. The expected value of the square of the voltage is identical to the expected value of the power given in eq. 2. The variance of the voltage is given by

$$V_V = E_P - (E_V)^2 = 1 + a^2 - ((2/\pi)(1+a)\ E(4a/(1+a)^2))^2 \qquad (7)$$

Again a simple approximate expression for the standard deviation of the voltage errors is desirable for computational purposes. A series approximation is given by

$$\sigma_V \approx (a/\sqrt{2})\,(1-3a^2/16)^{1/2} \qquad (8)$$

C. Phase Statistics

The error in the phase, referring to Fig. 1, is given by

$$\epsilon = \tan^{-1}(a\sin\alpha/(1+a\cos\alpha) \quad (9)$$

The mean phase error can be shown to equal 0 by observing the symmetry of the integrand. The variance of the phase error can be evaluated from

$$V_\varphi = (1/2\pi)\int_0^{2\pi}(\epsilon)^2\, d\alpha \quad (10)$$

This integral is difficult to evaluate in closed form. If the integrand is expanded in a Taylor's series, the following expression results

$$V_\varphi \approx a^2/2 + a^4/8 + \ldots \quad (11)$$

The standard deviation of the phase error is obtained from the square root of this expression.

D. Numerical Results

Numerical values for the preceding analysis are presented and the numerical accuracy of the series approximations is quantified. For convenience, the expressions for the statistics are gathered in Table I, where the $\approx$ signs indicate the series approximations used in the comparison. The error statistics for power and voltage are also plotted in their logarithmic form. The peak-to-peak error fluctuations, the rms error spread about the mean value and the mean error values are presented in Figs. 2 and 3 for power and voltage error statistics, respectively. Similarly, the peak-to-peak and rms statistics for phase errors are presented in Fig. 4.

The numerical accuracy for the series approximations is determined in the following way. The value a = 0.5 (-6 dB) was used as a reference level for the series approximations. The statistics for the voltage errors were computed from the elliptic integral values in the IMSL subroutine, and compared with the series expression. The errors in the series expressions were 1.6% and 0.1% for the mean and standard deviation of the voltage errors, respectively. Since the phase errors statistics cannot be expressed in closed form, the accuracy of the series approximation was compared to a numerical integration for the phase error; the error in the series approximation in this case was 0.5%. The accuracy of the series expressions improves for $a < 0.5$.

The comparison between the statistical values derived for the coherent error case with Gaussian statistics was done in the following manner. In contrast with zero-mean Gaussian statistics, the power and voltage error statistics have non-zero mean values. The standard deviations for Gaussian statistics were derived by assuming the peak-to-peak variations of the power, voltage and phase values represented $\pm 5\sigma$ variations, a common assumption. The resulting values are presented in Table I; in all cases, the standard deviations projected on the basis of Gaussian statistics are significantly lower than those derived for the coherent error case.

CONCLUSIONS

The error statistics for coherently related components have been derived. In contrast with values projected from Gaussian statistics, non-zero mean errors occur for power and voltage errors, and the standard deviation values are significantly higher. More details on the numerical accuracy and example applications may be found in Ref. 2.

REFERENCES

1. D. Middleton, An Introduction to Statistical Communication Theory, McGraw-Hill (1960), Chaps 7 and 9

2. R. B. Dybdal and R. H. Ott, "Coherent RF Error Statistics," Aerospace Tech Rept TR-0086(6925-05)-1

TABLE I ERROR STATISTICS

	POWER	VOLTAGE	PHASE
Mean	a^2	$\approx a^2/4$	0
Standard Deviation	$\sqrt{2}\ a$	$\approx a/\sqrt{2}$	$\approx a/\sqrt{2}$
Gaussian ($\pm 5\sigma$) Standard Deviation	$\approx 0.4a$	$\approx 0.2a$	$\approx 0.2a$

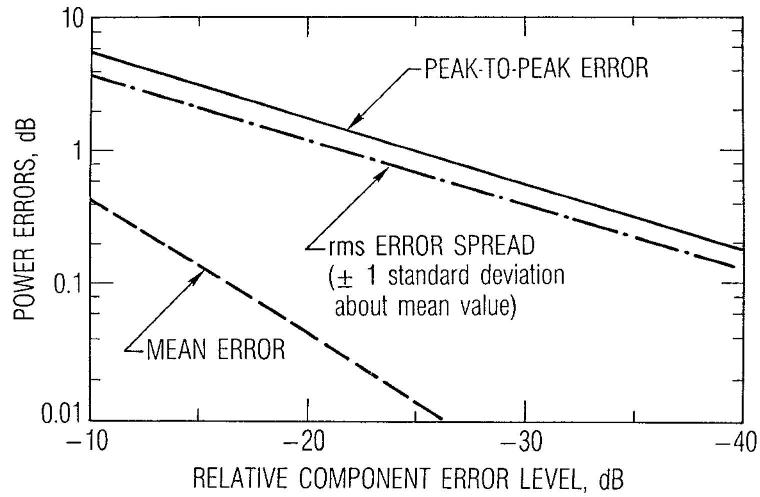

Fig. 2 Power Error Statistics

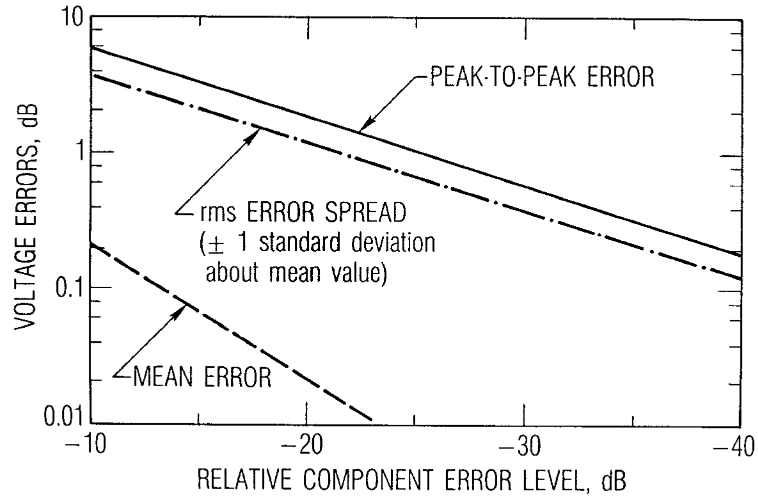

Fig. 3 Voltage Error Statistics

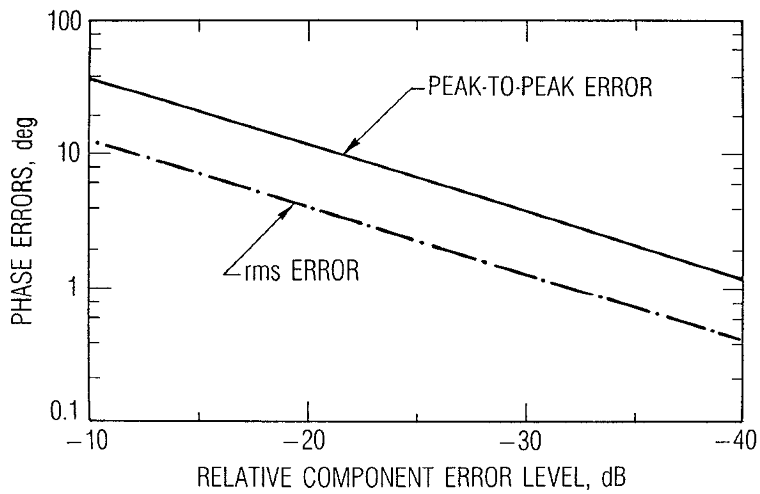

Fig. 4 Phase Error Statistics